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We show that two qubits can be entangled by local interactions with an entangled two-mode continuous variable state. This is illustrated by the evolution of two two-level atoms interacting with a two-mode squeezed state. Two modes of the squeezed field are injected respectively into two spatially separate cavities and the atoms are then sent into the cavities to resonantly interact with the cavity field. We find that the atoms may be entangled even by a two-mode squeezed state which has been decohered while penetrating into the cavity.

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I. INTRODUCTION

Entanglement is one of the important ingredients in current development of quantum information processing. If the density operator of a bipartite system, $\hat{\rho}$, is not represented by a convex sum of product states, i.e. $\hat{\rho} \neq \sum_i \hat{\rho}_a(i) \otimes \hat{\rho}_b(i)$ where $\hat{\rho}_a$ and $\hat{\rho}_b$ are density operators for subsystems a and b , the system is said to be entangled. For a 2×2 -dimensional space, Peres and Horodecki et al. [1] found a criterion of entanglement: When the partial transposition of its density matrix has a negative eigenvalue, the bipartite system is entangled. Entanglement is a purely quantum nature and it sometimes presents a contrast to nonlocality (see [2] for further discussions). Even though the two concepts coincide for pure bipartite states [3], when mixed states are concerned, there may be mixed entangled states which do not show nonlocality [4].

In this paper, we are interested in a possibility of entangling two remote qubits using a bipartite continuous variable state. This is illustrated by the evolution of two two-level atoms interacting with a two-mode squeezed state as shown in Fig. 1. Assume that two modes of the squeezed field are injected respectively into two spatially separate cavities. The atoms are then sent into the cavities to resonantly interact with the cavity field. A high- Q cavity is assumed to consider the Jaynes-Cummings type interaction between the atom and cavity field [5].

A pair of atoms can be prepared in an entangled state using the atom-field interaction in a high- Q cavity. The interaction of a single two-level atom with a cavity field brings about entanglement of the atom and the cavity field [6]. If the atom does not decay into other internal states after it comes out from the cavity, the entanglement will survive and it can be transferred to a second

atom interacting with the cavity field. The violation of Bell's inequality can be tested by the local measurement of atomic states.

There are proposals to entangle fields in two spatially separated cavities using the atom-field interaction [7,8]. A two-level atom in its excited state passes sequentially through two resonant single-mode vacuum cavities and is found to be in its ground state after the second-cavity interaction. If the interaction with the first cavity is equivalent to a $\pi/2$ vacuum pulse and the second-cavity interaction is to a π pulse then the atom could have deposited a photon either in the first cavity or in the second so that the final state $|\Psi_f\rangle$ of the two cavity field is [7]

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + e^{i\varphi}|0,1\rangle) \quad (1)$$

where $|1,0\rangle$ denotes one photon in the first cavity and none in the second, and $|0,1\rangle$ vice versa and the nonlocality of the cavity fields can also be tested using local parity measurement [9]. Using the entangled cavity field (1), an unknown atomic quantum state can also be teleported [10].

In quantum algorithm, the entangled state (1) of $\phi = 0$ is generated by the action of a controlled-NOT gate if the two input particles are in $|0\rangle(|0\rangle + |1\rangle)/\sqrt{2}$. Type-II [11] or type-I [12] parametric down-conversion may produce entangled states. By putting one photon state into a beam splitter, an entangled state is generated in the two output ports [13]. The beam splitter may also be used to produce a continuous-variable entangled state [14]. By a non-degenerate optical parametric amplifier a two-mode squeezed state, which is a continuous-variable entangled state, is generated [15].

We show the initial preparation of the cavity in Section II and the evolution of atoms and field in Section III. The evolution of entanglement between two atoms is studied in Section IV.

II. CAVITY FIELD

The two-mode squeezed state [17], which is generated using a non-degenerate optical parametric amplifier, is an entangled continuous-variable state. Maximal entanglement is not possible to produce for two-mode squeezed states because such a state implies infinite energy. The action of the two-mode squeeze operator,

$\hat{S}(s) = \exp(-s\hat{a}\hat{b} + s\hat{a}^\dagger\hat{b}^\dagger)$ on the vacuum $|0, 0\rangle$ produces the two-mode squeezed state

$$|\psi_F\rangle = \frac{1}{\cosh s} \sum_{n=0}^{\infty} (\tanh s)^n |n, n\rangle \quad (2)$$

where s is the squeezing parameter. Here \hat{a} and \hat{b} (\hat{a}^\dagger and \hat{b}^\dagger) are bosonic annihilation (creation) operators. The amount of entanglement of the two-mode squeezed state is linearly proportional to the squeezing parameter [16].

Each mode is injected to the cavity which is initially in the vacuum state as shown in Fig. 1. A complete quantum-mechanical picture of injecting an external field to a high-Q cavity is gained by modelling this operation with beam splitters [18]. For example, injecting $\hat{\rho}_f$ into a cavity initially in the vacuum, the cavity field becomes

$$\hat{\rho}_c = \text{Tr}_f \hat{B}(\theta) \hat{\rho}_f |0\rangle_c \langle 0| \hat{B}^\dagger(\theta) \quad (3)$$

where Tr_f is tracing over the mode f and the coupling between the external field and cavity is determined by the beam splitter operator $\hat{B}(\theta) = \exp[\frac{\theta}{2}(\hat{c}\hat{f}^\dagger - \hat{c}^\dagger\hat{f})]$ with \hat{c} and \hat{f} being annihilation operators respectively for the cavity and external fields and \hat{c}^\dagger and \hat{f}^\dagger are their hermitian conjugates. The reflection coefficient $r = \cos(\theta/2)$ of the beam splitter is determined by the coupling between the external field and cavity.

Extending Eq.(3) to coupling of the two-mode squeezed state (2) with two independent cavities initially in their vacuum states, we obtain the density operator for the cavity field, after injecting the squeezed state:

$$\hat{\rho}_c = \left(\frac{1}{\cosh s}\right)^2 \sum_{n,m=0}^{\infty} \sum_{k,l=0}^{\min[n,m]} (\tanh s)^{n+m} G_{kl}^{nm}(\theta) \quad (4)$$

$$\times |n-k, n-l\rangle \langle m-k, m-l|$$

where $G_{kl}^{nm}(\theta) = C_k^n(\theta) C_l^m(\theta) C_l^n(\theta) C_k^m(\theta)$ with

$$C_k^n(\theta) = \sqrt{\frac{n!}{k!(n-k)!}} \cos^k \frac{\theta}{2} \sin^{n-k} \frac{\theta}{2}. \quad (5)$$

The density matrix $\hat{\rho}_c$ represents the composite system of the cavity fields in a mixed state. It has been shown that the mixed state $\hat{\rho}_c$ in Eq.(4) is never separable regardless of θ using the separability criterion for a Gaussian field [19,20].

III. ATOM-FIELD INTERACTION

The Jaynes-Cummings model [5] is one of the most studied models in foundations of quantum mechanics. It consists of a two-level atom resonantly coupled to a single-mode radiation field inside a cavity. The importance of the Jaynes-Cummings model is many-fold. This exactly soluble simple model may serve a basis for more

complicated systems and it manifests interesting quantum nature of the field and atom. This model can also be experimentally realisable [21].

The Hamiltonian for the Jaynes-Cummings model in the interaction picture is written in atomic bases $\{|e\rangle, |g\rangle\}$ [5]

$$\hat{H}_I = \begin{pmatrix} 0 & \lambda \hat{c} \\ \lambda \hat{c}^\dagger & 0 \end{pmatrix} \quad (6)$$

where λ is the coupling coefficient between the atom and field. The corresponding unitary evolution operator $\hat{U} = \exp(-i\hat{H}_I t)$ can be written in the following form [22].

$$\hat{U}(t) = \begin{pmatrix} \cos \hat{\Omega}_{n+1} t & -i\lambda \hat{a} \frac{\sin \hat{\Omega}_n t}{\hat{\Omega}_n} \\ -i\lambda \hat{a}^\dagger \frac{\sin \hat{\Omega}_{n+1} t}{\hat{\Omega}_{n+1}} & \cos \hat{\Omega}_n t \end{pmatrix} \quad (7)$$

where the Rabi-frequency operator, $\hat{\Omega}_n = \lambda \sqrt{\hat{a}^\dagger \hat{a}}$.

In our model, there are two atoms residing in two separate cavities. The total unitary operator, thus, can be constructed by a direct product of two unitary operators representing evolutions of two independent atom-field interactions: $\hat{U}_T(t) = \hat{U}^A(t) \otimes \hat{U}^B(t)$ where A and B are the labels for two cavities. For simplicity, we assume that the atoms undergo the same evolution with the same atom-field coupling constants and interaction times.

IV. ENTANGLEMENT OF TWO ATOMS

We assume that the two atoms are prepared in a separable pure state $\hat{\rho}_a$ and the field is in the mixed two-mode squeezed state (4). The composite system of the atoms and field evolves unitarily and is represented by the density operator $\hat{\rho}(t)$ at time t :

$$\hat{\rho}(t) = \hat{U}_T(t) \hat{\rho}_a \otimes \hat{\rho}_c \hat{U}_T^\dagger(t). \quad (8)$$

Because we are interested in entanglement between the atoms, we trace over the field variables and find the time-dependent density operator $\hat{\rho}_a(t)$ for the two atoms:

$$\hat{\rho}_a(t) = \text{Tr}_c \{\hat{\rho}(t)\} = \begin{pmatrix} A(t) & 0 & 0 & E(t) \\ 0 & B(t) & 0 & 0 \\ 0 & 0 & C(t) & 0 \\ E(t) & 0 & 0 & D(t) \end{pmatrix} \quad (9)$$

where the matrix basis is chosen as $\{|e, e\rangle, |e, g\rangle, |g, e\rangle, |g, g\rangle\}$ and $A(t), B(t), C(t), D(t)$, and $E(t)$ are the functions of interaction time, which are determined by the initial state of the atoms. For example, if the atoms are initially in $|g, g\rangle$ the functions are given by

$$A(t) = \sum_{n=0}^{\infty} \sum_{k,l=0}^n K_{kl}^{nn}(\theta, s) \sin^2(\lambda t \sqrt{n-k}) \sin^2(\lambda t \sqrt{n-l})$$

$$\begin{aligned}
B(t) &= \sum_{n=0}^{\infty} \sum_{k,l=0}^n K_{kl}^{nn}(\theta, s) \sin^2(\lambda t \sqrt{n-k}) \cos^2(\lambda t \sqrt{n-l}) \\
C(t) &= \sum_{n=0}^{\infty} \sum_{k,l=0}^n K_{kl}^{nn}(\theta, s) \cos^2(\lambda t \sqrt{n-k}) \sin^2(\lambda t \sqrt{n-l}) \\
D(t) &= \sum_{n=0}^{\infty} \sum_{k,l=0}^n K_{kl}^{nn}(\theta, s) \cos^2(\lambda t \sqrt{n-k}) \cos^2(\lambda t \sqrt{n-l}) \\
E(t) &= \sum_{n=0}^{\infty} \sum_{k,l=0}^n K_{kl}^{n+1n}(\theta, s) \sin(\lambda t \sqrt{n-k+1}) \cos(\lambda t \sqrt{n-k}) \\
&\quad \times \sin(\lambda t \sqrt{n-l+1}) \cos(\lambda t \sqrt{n-l})
\end{aligned} \tag{10}$$

where

$$K_{kl}^{nm}(\theta, s) = \frac{(\tanh s)^{n+m}}{(\cosh s)^2} G_{kl}^{nm}(\theta). \tag{11}$$

For the case that the atoms are initially prepared in $|e, e\rangle$ the functions are modified from Eqs.(10). In this case, the functions can be obtained by switching $\sin \leftrightarrow \cos$ and $k, l \rightarrow k+1, l+1$ in Eqs.(10).

The density operator $\hat{\rho}_a(t)$ characterizes the atoms inside the cavity not in a pure state for $t > 0$. There are several ways to quantify a degree of entanglement for a mixed state such as the quantum relative entropy [23], the entanglement formation and the measure defined by negative eigenvalues for the partial transposition of the density operator [24]. Among the possible entanglement measure for a mixed state we take the measure of negative eigenvalues for the partial transposition of the density operator because the calculation is simple.

The partial transposition of $\hat{\rho}_a(t)$ has four eigenvalues one of which may be negative. The entanglement measure $\mathcal{E}(\hat{\rho})$ is defined as multiplying the factor -2 to the sum of negative eigenvalues. The entanglement measure then ensures the scale between 0 and 1 and monotonously increases as entanglement grows. The entanglement measure for the atoms inside the cavities are given by the function of interaction time as

$$\mathcal{E}[\hat{\rho}_a(t)] = \sqrt{[B(t) - C(t)]^2 + 4E(t)^2} - B(t) - C(t). \tag{12}$$

The measure of entanglement in Eq.(12) is not only the function of interaction time but also the function of squeezing parameter s . It is obvious that entanglement does not appear when the squeezing parameter $s = 0$ because the two-mode squeezed state becomes separable and entanglement is not generated only by local operations. For $s \neq 0$, we found that entanglement is produced between the atoms inside the cavities regardless of s when the atoms are initially prepared in $|g, g\rangle$ and $r = 0$.

In Fig.2, entanglement between two atoms is plotted against the squeezing parameter s at a specific interaction time $\lambda t = 11$. (We have chosen this time because the evolution of entanglement shows local maximum at this

time as shown later in Fig. 3.) The figure is presented for the three different values of the reflection coefficient r . It shows that the amount of entanglement between the two atoms (qubits) is not a simple increasing function of s and even decreases as s gets larger after it peaks at $s \approx 0.65$.

The two-mode squeezed state is represented as a superposition of the number states $|n, n\rangle$ as shown in Eq.(2). The two qubits which are in the $\mathcal{H}_2 \otimes \mathcal{H}_2$ are coupled with $|n, n\rangle, |n+1, n\rangle, |n, n+1\rangle$ and $|n+1, n+1\rangle$ of continuous variables at the Rabi frequency $\Omega_n = \lambda\sqrt{n}$. Because the amplitude $\tanh s$ in Eq.(2) increases with the squeezing parameter s , the terms of higher excitation, *i.e.* large n , contribute more to the atom-field interaction. The entanglement transfer is then cancelled and approaches to zero for $s \rightarrow \infty$. It means the maximally entangled continuous variable system does not generate maximally entangled qubits by the unitary transformation we have in this paper.

In Fig.3, we plot the time evolution of entanglement for $s = 0.65$ for the different values of the reflection coefficient r when the atoms are initially prepared in $|e, e\rangle$ and in $|g, g\rangle$. The amount of entanglement decreases as the reflection coefficient r is increased while its dynamics exhibits the same qualitative behaviors. The reflection coefficient r represents the degree of mixedness of the two-mode squeezed state. As r gets larger, the amplitude of the cavity field gets smaller. However, it is interesting to note that when atoms are initially in $|g, g\rangle$, even with r as large as 0.99, the atoms may be entangled. Entanglement is maximised when the cavities are initially prepared with a pure two-mode squeezed state ($r = 0$).

It is seen that entanglement evolves depending on the initial preparation of atoms. When the atoms are initially in $|e, e\rangle$, entanglement does not appear at the early stage of the interaction and undergoes fast oscillations. Compared with the $|e, e\rangle$ preparation, the initial $|g, g\rangle$ state produces more entanglement from the early stage of interaction. Because of extra photons which are emitted by atoms initially in $|e, e\rangle$, the atom-field interaction becomes more complicated and brings about faster oscillations in entanglement.

V. REMARKS

We consider the entanglement of two two-level atoms by local interaction with the two-mode squeezed field. We found that the entanglement is maximised when the two-mode squeezed field is pure. As far as the two-mode squeezed state is pure, the atoms show entanglement during their evolution but the entanglement goes to zero as $s \rightarrow \infty$. The dynamics of entanglement does not change its time dependence while the overall amplitude is lowered when the initial squeezed field becomes mixed. We found that the entanglement depends on initial preparation of atoms.

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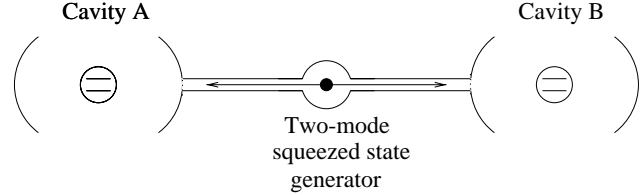


FIG. 1. Each mode of the two mode squeezed state is injected into the high Q cavities A and B

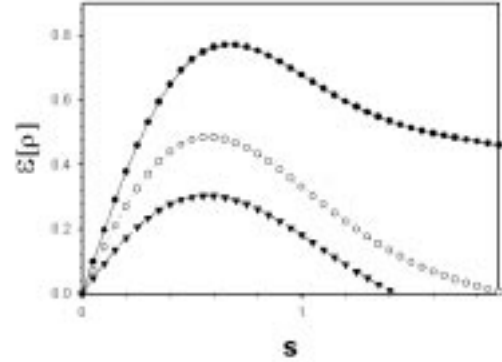


FIG. 2. Entanglement between two atoms versus squeezing parameter s at the interaction time $\lambda t = 11$. The cavity fields are initially in the pure state $r = 0$ (solid circle), mixed state $r = 0.25$ (open circle) and $r = 0.7$ (triangle). The atoms are initially in $|g, g\rangle$.

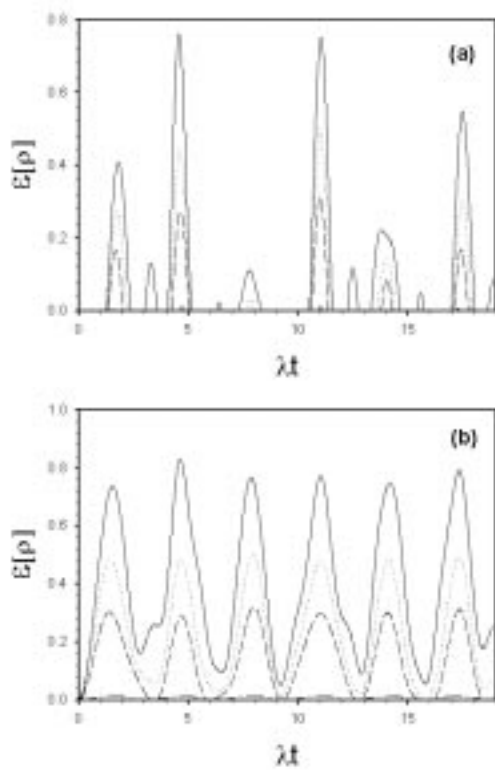


FIG. 3. The time evolution of entanglement between the two atoms is plotted as the function of dimensionless time λt . The cavity reflection parameters are chosen as $r = 0$ (solid), $r = 0.25$ (dotted), $r = 0.7$ (dashed) and $r = 0.99$ (dashed-dot). The squeezing parameter is $s = 0.65$. Initial atomic states are prepared in $|e, e\rangle$ (a) and $|g, g\rangle$ (b).